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The First Mathematically Correct Life Annuity Valuation Formula*

The story of the first actuarially correct specification of a life annuity revolves around Jan (Johan) de Witt (1625-1672), Jan Hudde (1628-1704), and Christiaan Huygens (1629-1695). All were Dutch and mathematics students of Frans van Schooten (1615-1660). De Witt was born into a politically prominent family, was a naval officer who fought against the British on the high seas, and was the grand pensionary (prime minister) of Holland at age 28. Hudde was head of the Dutch admiralty, a mayor and long-term burgomaster (a high profile political post) of Amsterdam, and a political opponent of de Witt. Huygens was a prominent mathematician and contributor to probability theory and considered by some to be the leading natural philosopher immediately prior to the age of Isaac Newton.

In 1656 Huygens wrote a short treatise in Dutch on probability theory. Van Schooten convinced Huygens to publish his treatise as an appendix at the end of a book that van Schooten was preparing. Van Schooten himself prepared a Latin translation of Huygens work and published it, under Huygens name, in his book in 1657. The following proposition, published in the van Schooten appendix, was among Huygens contributions to probability theory (Hald, 2003):

> If the number of chances of getting a is p, and the number of chances of getting b is q, assuming always that any chance occurs equally easily, then this is worth $(pa+qb)/(p+q)$.

We might rewrite Huygens last expression as

$$a\left[\frac{p}{p+q}\right]+b\left[\frac{q}{p+q}\right];$$

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and, in more modern terminology, let X be a random variable that takes on the value a with probability $p/(p+q)$ and the value b with probability $q/(p+q)$. Then the expected value of the random variable X is

$$E[X] = a\left[\frac{p}{p+q}\right] + b\left[\frac{q}{p+q}\right]. \tag{1}$$

Thus, the concept of *mathematical expectation* or *expected value* was born, or at least put in print for the first time, approximately 350 years ago.[1]

In 1669 Huygens and his brother Ludwig engaged in correspondence with each other that clearly applied the idea of expected value to the calculation of life expectancy.[2] They focused on the distribution of deaths; or, looked at from another point of view, what might be termed the distribution of remaining life or the distribution of additional years of life given a person age x. Let l_x denote the number of survivors at age x, and $d_x = l_x - l_{x+1}$ denotes the number who die between ages x and x+1. Then the probability of living t years (not less and not more) beyond age x is given by d_{x+t}/l_x because to live t years (no more and no less) requires that one survives t years but not t+1 years from age x. That is, death must occur between ages x+t and x+t+1. Let T be a random variable for remaining years of life, then *life expectancy*, from the Huygens brothers' point of view, is the expected value of T and can be written as

$$E(T) = \sum_{t=0}^{\omega-x-1} t(d_{x+t}/l_x) \tag{2}$$

where ω is the youngest age at which there are no survivors $(l_\omega = 0)$.[3]

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[1]To Huygens, the mathematical expectation was the fair price of a gamble. Hacking (1975) indicates that Huygens was concerned with determining a fair value or price of a gamble, and the answer to that question was the expected value of the gamble. Stigler (1999) has the following translation of Huygens: If the number of chances leading to a is p, and the number of chances leading to b is q, and all chances are equally likely, then the expectation is valued at $(pa+qb)/(p+q)$.

[2]Ludwig Huygens may have been the first to make a life expectancy calculation. This correspondence also reveals that the Huygens brothers clearly understood the difference between the expected value and the median value of a random variable. They distinguished between the average number of additional years of life and the number of years until half of a population had died and half still lived.

[3]Today, we would call this the *curtate life expectancy*. That is, a person must live a full year for that year to count in life expectancy. More typically we assume that deaths occur uniformly throughout a year; and therefore people who die within a year will live, on average, one-half of the year in which they die. The complete life expectancy becomes the cu-

In the mid-seventeenth century, sales of life annuities were a common source of local and national finance in Holland and some other European countries. Standard practice of the day dictated selling annuities at one price regardless of the age of the nominee (*i.e.*, the person on whose continued life annuity payments depend). Most nominees, as would be predicted when the price is unrelated to age, were very young. Hudde's data showed that 80% of all nominees were under age 20 and half were less than 10 years old (Poitras, 2000). As prime minister, de Witt dealt with the financial consequences of Anglo-Dutch wars and in 1671 was preparing for war with France. In short, the Dutch government needed money and de Witt proposed selling more life annuities to generate funds; but the correct mathematical formula for pricing annuities was heretofore unknown. In 1671 de Witt issued a report which showed the correct actuarial valuation of a life annuity, a report which demonstrated for the first time how to correctly integrate compound interest and mortality probabilities into the valuation of a life annuity. Hudde read de Witt's report and attested to the mathematical appropriateness of his methods.

Here is de Witt's method. Let $a_{\overline{t}|}$ denote an ordinary annuity certain (*i.e.*, an annuity immediate) defined in the usual way as

(3) $$a_{\overline{t}|} = \sum_{j=1}^{t} (1+i)^{-j}$$

where i denotes the interest rate. Now, think of this annuity certain as a random variable by allowing the term t to vary from one to an arbitrarily large number. Define T to be a random variable measuring remaining life time as in formula (2) above, then the random variable $a_{\overline{T}|}$ takes on the values $a_{\overline{1}|}$, $a_{\overline{2}|}$, $a_{\overline{3}|}$, … with probabilities d_{x+1}/l_x, d_{x+2}/l_x, d_{x+3}/l_x,…. That is, a nominee gets the annuity certain $a_{\overline{1}|}$ if he survives one year but not two years (which occurs with probability d_{x+1}/l_x), a nominee gets the annuity certain $a_{\overline{2}|}$ if he survives two years but not three years (which occurs with probability d_{x+2}/l_x), and so on. Thus, de Witt's formula (de Witt, 1671 and Hendricks, 1852, 1853) for a life annuity becomes[4]

(4) $$E(a_{\overline{T}|}) = \sum_{t=1}^{\omega-x-1} a_{\overline{t}|}(d_{x+t}/l_x) .$$

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rate expectancy plus .5. Using the typical notation, complete life expectancy for a life age x is $\dot{e}_x = .5 + E(T)$, where $E(T)$ is defined in formula (2).

[4]We can imagine a lead term in this sum to be $a_{\overline{0}|}(d_x/l_x)$; but we define $a_{\overline{0}|} = 0$, meaning that nothing is paid if a nominee dies between age x and x+1. A nominee must survive one year to get the first annuity payment.

At this point, we notice the similarity between de Witt's formula (4) and the Huygens brothers' formula (2). One simply replaces the time variable t in formula (2) with the annuity certain variable $a_{\overline{t|}}$ and formula (4) emerges (Hald, 2003). In modern terminology, we would say that de Witt defined an annuity certain with a random term as a random variable and specified its probability distribution. Formula (4) is insightful because it says that the value of a life annuity is the *expected present value* of the annuity certain random variable. In modern times, we often speak of the *present value* of a life annuity; but de Witt's formulation was clearly an *expected present value*. His approach also enables us to compute higher order moments and any other characteristic of the $a_{\overline{T|}}$ random variable (Haberman and Sibbett, 1995).

De Witt showed that the correct price of a life annuity for a three-year old nominee to be16 florins (florins being the standard of value of the day) when Holland was selling such annuities for 14 florins, using the interest rate of 4% per annum. After discovering formula (4), de Witt's biggest practical problem was obtaining accurate mortality probabilities. He assumed level mortality between ages three and 53 and declining mortality thereafter for age groups 53-63, 63-73, and 73-80 (Hald, 2003 and Poitras, 2000). Subsequent correspondence between de Witt and Hudde discussed problems like annuities on joint lives, what mortality probabilities should be used in life annuity calculations, and self selection of nominees – important problems to this day. In 1672 Holland started selling life annuities at prices that were inversely related to age. However, prices were below those calculated by de Witt and Hudde.

As an epilog, we note that after 1671 Hudde and Huygens continued their careers and eventually died of natural causes. De Witt was not so fortune. France invaded Holland in 1672, and de Witt resigned as prime minister. A mob, supporting the Prince of Orange, murdered him and his brother and then mutilated their bodies. The gruesome deed is captured in a painting (exhibited at the Rijksmuseum in Amsterdam) entitled *The Bodies of the Brothers de Witt, Martyrs of the Republic* by Jan de Baen showing their eviscerated bodies hanging upside down. A statue of de Witt was erected in 1918 in The Hague on the spot of his murder; the inscription honors him for his contributions to politics, the navy, state finance, and mathematics. The statue was dedicated by Queen Wilhelmina who, somewhat ironically, was from the House of Orange-Nassau.

— James E. Ciecka

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