Part 1.   Algebra Review

A.   The Number System

1.   Real Numbers = numbers that we encounter everyday during a normal course of life

   i. Natural numbers = the numbers that we often use to count items → counting trees, apples, bananas, etc.: 1, 2, 3, 4, …

      1. odd numbers: 1, 3, 5, …
      2. even numbers: 2, 4, 6, …

   ii. Integers = whole numbers without a decimal point: $0, \pm1, \pm2, \pm3, \pm4, \ldots$

      1. positive integers: 1, 2, 3, 4, …
      2. negative integers: -1, -2, -3, -4, …

   iii. Rational numbers = numbers that can be expressed as a fraction of integers such as a/b where both a and b are integers

      1. finite decimal fractions: 1/2, 2/5, etc.
      2. (recurring or periodic) infinite decimal fractions: 1/3, 2/9, etc.

   iv. Irrational Numbers = numbers that can NOT be expressed as a fraction of integers such as a/b where both a and b are integers = nonrecurring infinite decimal fractions:

$$\sqrt{2}, \sqrt[3]{5}, 7^{\sqrt{3}}, \text{pi}(\pi), \text{etc.}$$

   v. Undefined fractions:

      1. any number that is divided by a zero such as p/0 where p is any number

      2. a zero divided by a zero = 0/0

      3. an infinity divided by an infinity = $\dfrac{\infty}{\infty}$

vi. Defined fractions:

1. a zero divided by an infinity $= \dfrac{0}{\infty} = 0$

2. a one divided by a very small number $=$
   $$\dfrac{1}{0.0000000001} = \dfrac{1}{10^{-10}} = 10^{10} = 10,000,000,000 \approx \text{a very}$$
   large number such as ∞

3. a scientific notion $\rightarrow$ the use of exponent

   $2.345E+2 = 2.345 \times 10^2 = 234.5$

   $2.345E+6 = 2.345 \times 10^6 = 2,345,000$

   $2.345E-2 = 2.345 \times 10^{-2} =$
   $$2.345 \cdot \dfrac{1}{10^2} = 2.345 \cdot \dfrac{1}{100} = 0.02345$$

   $2.345E-6 = 2.345 \times 10^{-6} =$
   $$2.345 \cdot \dfrac{1}{10^6} = 2.345 \cdot \dfrac{1}{1,000,000} = 0.000002345$$

2. Imaginary Numbers = numbers that are not easily encountered and recognized on a day-to-day basis.

$$i = \sqrt{-1} \qquad\qquad \sqrt{-2} = \sqrt{2}i = i\sqrt{2}$$

$$\sqrt{-4} = 2i \qquad\qquad (5i)^2 = -25$$

B. Rules of Algebra

1. $a + b = b + a$ 2. $ab = ba$

3. $aa^{-1} = 1$ for $a \neq 0$ 4. $a(b + c) = ab + ac$

5. $a + (-a) = 0$ 6. $(-a)b = a(-b) = -ab$

7. $(a + b)^2 = a^2 + 2ab + b^2$ 8. $(a - b)^2 = a^2 - 2ab + b^2$

9. $(a + b)(a - b) = a^2 - b^2$

10. $\dfrac{-a}{-b} = (\text{-a})/(\text{-b}) = a/b$

11. $\dfrac{-a}{b} = \dfrac{a}{-b} = -1 \cdot \dfrac{a}{b}$

12. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$

13. $a + \dfrac{b}{c} = \dfrac{ac + b}{c}$

14. $a \times \dfrac{b}{c} = a \cdot \dfrac{b}{c} = \dfrac{ab}{c}$

15. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

16. $a^{1/2} = a^{0.5} = \sqrt{a}$ where $a \geq 0$

17. $a^{1/n} = \sqrt[n]{a}$ where $a \geq 0$

18. $\sqrt{ab} = \sqrt{a} * \sqrt{b}$

19. $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

20. $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

21. $\dfrac{\sqrt{a}}{b} \neq \sqrt{\dfrac{a}{b}}$

22. $a^n \cdot a^m = a^{n+m}$

23. $(a^n)^m = a^{nm}$

24. $a^{p/q} = (a^{1/q})^p = (a^p)^{1/q} = \sqrt[q]{a^p}$

C. Properties of Exponents

    i) $X^0 = 1$ →   Note that 0^0 = undefined

    ii) $\dfrac{1}{X^b} = X^{-b}$

    iii) $X^a * X^b = X^a \cdot X^b = X^a X^b = X^{a+b}$

    iv) $(X^a)^b = X^{a*b} = X^{a \cdot b} = X^{ab}$

    v) $\dfrac{X^a}{X^b} = X^a \cdot X^{-b} = X^{a-b}$

    vi) $(XY)^a = X^a * Y^a = X^a \cdot Y^a = X^a Y^a$

    vii) $\sqrt[n]{X} = X^{\frac{1}{n}} = X^{1/n}$

D.   Compound Interest

    1.   The Concept of Periodic Interest Rates

Assume that the annual percentage rate (APR) is $(r \cdot 100)\%$. That is, if an APR is 10%, then r = 0.1. Further, define FV = future value, PV = present value, and t = number of years to compound.

    i)   Annual compounding for t years

$$FV = PV \cdot (1+r)^t$$

    ii)   Semiannual compounding for t years

$$FV = PV \cdot (1+\frac{r}{2})^{2t}$$

    iii)   Quarterly compounding for t years

$$FV = PV \cdot (1+\frac{r}{4})^{4t}$$

    iv)   Monthly compounding for t years

$$FV = PV \cdot (1+\frac{r}{12})^{12t}$$

    v)   Weekly compounding for t years

$$FV = PV \cdot (1+\frac{r}{52})^{52t}$$

    vi)   Daily compounding for t years

$$FV = PV \cdot (1+\frac{r}{365})^{365t}$$

    vii)   Continuous compounding for t years

$$FV = PV \cdot e^{rt}$$

Examples>

Assume that $100 is deposited at an annual percentage rate (APR) of 12% for 1 year.

i)   Annual compounding → one 1-year deposit → 1 interest calculation

$$FV = PV \cdot (1+r)^t = \$100 \cdot (1+0.12)^1 = \$112.00$$

ii)   Semiannual compounding →two ½-year deposits → 2 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{2})^{2t} = \$100 \cdot (1+\frac{0.12}{2})^{2 \cdot 1} = \$100 \cdot (1+0.06)^2 = \$112.36$$

iii)   Quarterly compounding →four ¼-year deposits → 4 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{4})^{4t} = \$100 \cdot (1+\frac{0.12}{4})^{4 \cdot 1} = \$100 \cdot (1+0.03)^4 = \$112.55$$

iv)   Monthly compounding → twelve 1/12-year deposits → 12 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{12})^{12t} = \$100 \cdot (1+\frac{0.12}{12})^{12 \cdot 1} = \$100 \cdot (1+0.01)^{12} = \$112.68$$

v)   Weekly compounding →fifty-two 1/52-year deposits → 52 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{52})^{52t} = \$100 \cdot (1+\frac{0.12}{52})^{52 \cdot 1} = \$100 \cdot (1+0.0023077)^{52} = \$112.73$$

vi)   Daily compounding →365 1/365-year deposits → 365 interest calculations in 1 year

$$FV = PV \cdot (1+\frac{r}{365})^{365t} = \$100 \cdot (1+\frac{0.12}{365})^{365 \cdot 1} = \$100 \cdot (1+0.000328767)^{365} = \$112.74$$

vii) Continuous compounding for 1 year → continuous interest calculations

$$FV = PV \cdot e^{rt} = \$100 \cdot e^{0.12 \cdot 1} = \$100 \cdot e^{0.12} = \$112.75$$

2. The Concept of Compounding

i) A growth factor, $(1+\frac{g}{100})^t$, for a growth rate of g %

$$FV = PV \cdot \left(1+\frac{g}{100}\right)^t$$

Example> If you deposit $100 in a bank for 2 years and earn (=grow) 12% per year, how much money will you have in your account at the end of the 2-year period? Assume annual compounding only.

Answer:

$$FV = PV \cdot (1+\frac{g}{100})^t = \$100 \cdot (1+\frac{12}{100})^2 = \$100 \cdot (1+0.12)^2 = \$125.44$$

ii) A discount factor, $(1+\frac{d}{100})^{-t}$, for a discount rate of d %:

$$PV = \frac{FV}{(1+\frac{d}{100})^t} = FV \cdot (1+\frac{d}{100})^{-t}$$

Example 1> If you received $125.44 after having deposited some money (X) in a bank for 2 years at an annual interest rate of 12%, how much money (X) did you deposit in your account 2 years ago? Identify the discount factor. Assume annual compounding only.

Answer:

$$PV = FV /(1+\frac{g}{100})^t = \$125.44 /(1+\frac{12}{100})^2 = \$125.44 \cdot (1+0.12)^{-2} = \$100.00$$

Discount factor = $(1+0.12)^{-2} = 0.7971938$

Example 2>   If a borrower promises to pay you $146.41 at the end of the 4-th year and the current interest rate is 10%, how much are you willing to lend him/her now? Identify the discount factor. Assume annual compounding only.

Answer:

$$PV = FV/(1 + \frac{g}{100})^t = \$146.41/(1 + \frac{10}{100})^4 = \$146.41 \cdot (1 + 0.1)^{-4} = \$100.00$$

Discount factor $= (1 + 0.10)^{-4} = 0.683013455$

E.   Application to Annuity Calculation

Annuity Formulas:

$$FV = \frac{A \cdot [(1 + i)^n - 1]}{i}$$

$$PV = \frac{A \cdot [(1 + i)^n - 1]}{i \cdot (1 + i)^n}$$

where A= the fixed annuity amount; n = the number of periods; and i = a periodic interest rate. Of course, FV = the future (or final or terminal) value and PV = the present (or current) value.

Examples

1. If you obtain a 30 year mortgage loan of $100,000 at an annual percentage rate (APR) of 6%, what would be your monthly payment?

Answer:

$$100,000 = \frac{A \cdot [(1 + \frac{0.06}{12})^{12 \times 30} - 1]}{\frac{0.06}{12} \cdot (1 + \frac{0.06}{12})^{12 \times 30}}$$

Therefore, A=$599.55

2. If you invest \$1,000 a month in an account that is guaranteed to yield a 10% rate of return per year for 30 years (with a monthly compounding), what will be the balance at the end of the 30-year period?

Answer:

$$FV = \frac{\$1,000 \cdot [(1+\frac{0.1}{12})^{360} - 1]}{\frac{0.1}{12}} = \$2,260,487.92$$

3. If you are guaranteed of a 10% rate of return for 30 years, how much should you save and invest each month to accumulate \$1 million at the end of the 30-year period?

Answer:

$$\$1,000,000 = \frac{A \cdot [(1+\frac{0.1}{12})^{360} - 1]}{\frac{0.1}{12}}$$

Therefore, A=\$442.38

4. Suppose that you have saved up \$100,000 for your retirement. You expect that you can continuously earn 10% each year for your \$100,000. If you know that you are going to live for 15 additional years from the date of your retirement and that the balance of your retirement fund will be zero at the end of the 15-year period, how much can you withdraw to spend each month?

Answer:

$$\$100,000 = \frac{A \cdot [(1+\frac{0.1}{12})^{180} - 1]}{\frac{0.1}{12} \cdot (1+\frac{0.1}{12})^{180}}$$

Therefore, A=\$1,074.61

5. Assume the same situation as Problem 4 above, except that now you have to incorporate an annual inflation rate of 3%. What will be the possible monthly withdrawal, net of inflation?

Answer:

$$\$100{,}000 = \frac{A \cdot [(1 + \frac{0.1 - 0.03}{12})^{180} - 1]}{\frac{0.1 - 0.03}{12} \cdot (1 + \frac{0.1 - 0.03}{12})^{180}}$$

Therefore, A=$898.83

Note: Combining Answers to Problems 4 and 5, it means that you will be actually withdrawing $1,074.61 per month but its purchasing power will be equivalent to $898.83. This is because inflation only erodes the purchasing power; it does not reduce the actual amount received. If one goes through a professional financial planning, the financial planner will expand on this simple assumption to a more complex and realistic scenario.

6. Assuming only annual compounding, how long will it take to double your investment if you earn 10% per year?

   Answer:

   $$A \cdot (1 + 0.1)^x = 2A$$
   $$\therefore 1.1^x = 2$$
   $$\ln 1.1^x = \ln 2$$
   $$\therefore X = \frac{\ln 2}{\ln 1.1} = 7.2725 years$$

7. Assuming monthly compounding, how long will it take to double your investment if you earn 10% per year?

   Answer:

   $$A \cdot (1 + \frac{0.1}{12})^x = 2A$$
   $$\therefore 1.0083333^x = 2$$
   $$\ln 1.0083333^x = \ln 2$$
   $$\therefore X = \frac{\ln 2}{\ln 1.0083333} = 83.5 months = 6.96 years$$

8. Assume that you have a 30-year, $100,000 mortgage loan at an annual percentage rate (APR) of 6%. How long will it take you to pay off this loan if you pay off $1,000 a month?

Answer: Use the information on Answers to Problem 1 as follows:

$$100,000 = \frac{1,000 \cdot [(1+\frac{0.06}{12})^X - 1]}{\frac{0.06}{12} \cdot (1+\frac{0.06}{12})^X}$$

Therefore,

$$100,000 \cdot \frac{0.06}{12} \cdot (1+\frac{0.06}{12})^X = 1,000 \cdot [(1+\frac{0.06}{12})^X - 1]$$

$$500 \cdot (1+\frac{0.06}{12})^X = 1000 \cdot (1+\frac{0.06}{12})^X - 1000$$

$$500 \cdot (1.005)^X = 1000$$

$$X \cdot \ln 1.005 = \ln 2$$

$$X = \frac{\ln 2}{\ln 1.005} = 135.975 months = 11.58 years$$

E.   Inequalities

1.   If $a > 0$ and $b > 0$, then $(a+b) > 0$ and $ab > 0$

   If a=7 and b=5, then $(7+5) > 0$ and $(7)(5)>0$

2.   If $a > b$, then $(a-b) > 0$

   If a=7 and b=5, then $(7-5)>0$

3.   If $a > b$, then $(a+c) > (b+c)$ for all c

   If a=7 and b=5, then $(7+c) > (5+c)$ → 7>5

4.   If $a > b$ and $c > 0$, then $ac > bc$

   If a=7 and b=5 and c=3, then $(7)(3) > (5)(3)$ → 21 > 15

5.   If $a > b$ and $c < 0$, then $ac < bc$

If a=7 and b=5 and c= -3, then $(7)(-3) < (5)(-3)$ → -21<-15

F.   Absolute Values and Intervals

    1. $|X| = X$ if $X \geq 0$ and $|X| = -X$ if $X \leq 0$

    Examples>

        $|5| = +5$ or $-5 = \pm 5$

        $|+5| = +5 = 5$ and $|-5| = +5 = 5$

    2.   If $|X| \leq n$, then $-n \leq X \leq n$

    Examples>

        If $|X| \leq 5$, then $-5 \leq X \leq 5$

        If $|X-2| \leq 5$, then $-5 \leq X-2 \leq 5$ → $-5+2 \leq X \leq 5+2$ → $-3 \leq X \leq 7$

    3.   If $|X| > n$, then $X > n$ if $X > 0$ or $X < -n$ if $X < 0$

    Examples>

        If $|X| > 5$,

                then, $X > 5$ or $X < -5$.

        If $|X - 3| > 5$,

                then, $(X - 3) > 5$ → $X > 8$

                or $(X - 3) < -5$ → $X < -5 + 3$ → $X < -2$

G.   A System of Equations in Two Unknowns

        Given the following system of equations, solve for x and y.

                $3x + 2y = 13$
                $4y - 2x = 2$

    1.   Solution Method 1: The Substitution Method

(1) Rearrange the bottom equation for x as follows:

$$2x = 4y - 2$$
$$\therefore x = 2y - 1$$

(2) Substitute this x into the top equation as follows:

$$3(2y - 1) + 2y = 13$$
$$8y = 16$$
$$y = \frac{16}{8} = 2$$

(3) Substitute this y into any of the above equation for x value:

$$x = 2y - 1 = 2 \times 2 - 1 = 3$$

(4) Verify if the values of x and y satisfy the system of equations:

$$3x + 2y = 3 \times 3 + 2 \times 2 = 13$$
$$4y - 2x = 4 \times 2 - 2 \times 3 = 2$$

(5) Verification completed and solutions found.

2.   Solution Method 2: The Elimination Method

(1) Match up the variables as follows:

$$3x + 2y = 13$$
$$-2x + 4y = 2$$

(2) Multiply either of the two equations to find a common coefficient. (y is chosen and thus, the top equation is multiplied by 2 as follows:)

$$2 \times 3x + 2 \times 2y = 2 \times 13$$
$$6x + 4y = 26$$

(3) Subtract the bottom equation from the adjusted top equation and obtain:

$$6x + 4y = 26$$
$$-(-2x + 4y = 2)$$
$$\overline{\hspace{3cm}}$$

$$6x - (-2x) + 4y - 4y = 26 - 2$$
$$8x = 24$$
$$x = 3$$

(4) Substitute this x into any of the above equation for y value:

$$6x + 4y = 26$$
$$6 \times 3 + 4y = 26$$
$$4y = 26 - 18$$
$$y = 2$$

(5) Verify if the values of x and y satisfy the system of equations:

$$3x + 2y = 3 \times 3 + 2 \times 2 = 13$$
$$4y - 2x = 4 \times 2 - 2 \times 3 = 2$$

(6) Verification completed and solutions found.

3.   An Example

Suppose that you have $10 with which you can buy apples (A) and oranges (R). Also, assume that your bag can hold only 12 items – such as 12 apples, or 12 oranges, or some combination of apples and oranges. If the apple price is $1 and the orange price is $0.50, how many apples and oranges can you buy with your $10 and carry them home in your bag?

Answer:

The Substitution Method:

(1)   identify relevant information:

Budget Condition: A + 0.5R = 10
Bag-Size Condition: A + R = 12

(2)   convert the Bag-Size Condition as:

A = 12 – R

(3)   substitute A = 12 – R into the Budget Condition as:

(12 – R) + 0.5R = 10

$$-0.5R = -2 \quad \rightarrow \quad R=4$$

(4)   substitute R=r into (2) above and find:

$$A = 12 - 4 = 8$$

(5)   verify the answer of A=8 and R=4 by plugging them into the above two conditions as:

Budget condition: $8 + 0.5(4) = 10$
Bag-Size condition: $8 + 4 = 12$

Because both conditions are met, the answer is A=8 and R=4.

The Elimination Method:

(1)   identify relevant information:

Budget Condition: $A + 0.5R = 10$
Bag-Size Condition: $A + R = 12$

(2)   subtract the bottom equation from the top:

$$-0.5R = -2 \quad \rightarrow \quad R=4$$

(3)   plug this R=4 into either one of the two conditions above:

$$A + 0.5(4) = 10 \qquad \rightarrow A=8$$

Or $A + (4) = 12 \qquad \rightarrow A=8$

(4)   verify the answer of A=8 and R=4 by plugging them into the above two conditions as:

Budget condition: $8 + 0.5(4) = 10$
Bag-Size condition: $8 + 4 = 12$

Because both conditions are met, the answer is A=8 and R=4.