Mathematics Review
For
GSB 420

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I. Algebra Review

A. Solving Simultaneous Equations

Two equations with two unknowns

Supply: $Q^S = 75 + 3P$
Demand: $Q^D = 125 - 2P$

Solve for Equilibrium P and Q.

$75 + 3P = 125 - 2P$
$5P = 50$
$P = 10$

$Q = 75 + 3(10) = 105$ or $Q = 125 - 2(10) = 105$

B. Simplifying Algebraic Expressions

1. $x^0 = 1$

2. $x^{-a} = \dfrac{1}{x^a}$

3. $\dfrac{x^a}{x^b} = x^{a-b}$

4. $(x^a)^b = x^{a*b}$

5. $(x^a)*(x^b) = x^{a+b}$

C. Exponential and Logarithmic Functions

1. Exponential Functions

   a. Simple Exponential Function:
   $y = b^t$, has base *b* and exponent *t* and where b >1

   [graph]

   b. Generalized Exponential Function

$y = ab^{ct}$, where a and c are *compressing* and *extending* parameters. a pivots the function vertically and c extends it horizontally.

[graph]

c. Exponential Functions with base e (where e is the irrational number 2.71828 . . .)

$y = e^{t}$, or more generally,

$y = Ae^{rt}$

Applications of $y = Ae^{rt}$:

(1) Interest Compounding
(2) Instantaneous Rate of Growth (when we get to derivatives)

2. Logarithmic Functions

Natural Log

$y = e^{t}$ ⇔ $t = \log_e y$ (or $t = \ln y$)

Applications with Logs:

(1) Calculating Growth Rates
(2) Linearizing non-linear functions

Rules of Logarithms

(1) Log of a Product: $\ln(uv) = \ln(u) + \ln(v)$

(2) Log of a Quotient: $\ln(u/v) = \ln(u) - \ln(v)$

(3) Log of a Power: $\ln(u^{a}) = a\ln(u)$

[graph of y vs. t and t vs. y and ln$y$ vs. t]

II. Calculus –The Derivative

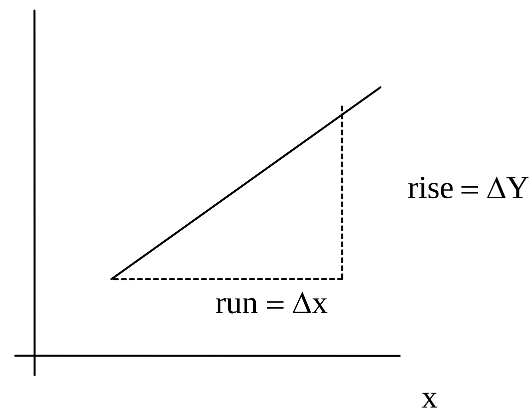
Concept: We wish to find the slope of a function. Knowledge of a slope is useful in providing a numerical measure of a relationship between two variables. This is a basis of causality and decision making.

The slope is defined as the rise over the run:

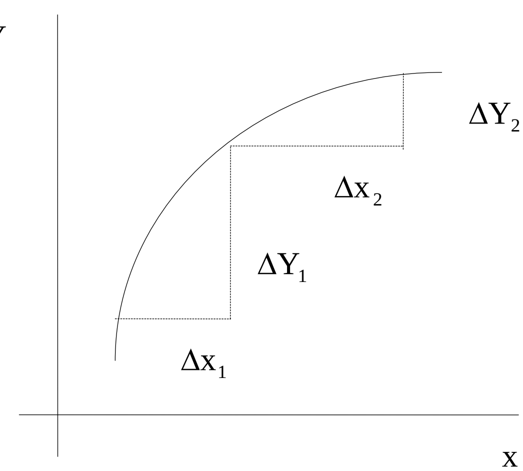
Given the function, $Y = f(x)$,

Slope = $\frac{\text{rise}}{\text{run}} = \frac{\Delta Y}{\Delta x}$

For a linear function the slope is a constant throughout the function and ….easy to calculate.



However, with a non-linear function the slope varies with x. However, we cannot measure the slop at a particular x because we need a change in x. If we try measuring the slope for a discrete change in x, we end up measuring it over an arc of the function. To measure the slope for any particular x we must make the change in x infinitesimally small.



We define the slope as:

$$\frac{\partial Y}{\partial x} = \lim_{\Delta x \to o} \frac{\Delta Y}{\Delta x}$$

That is; the derivative of Y with respect to x is defined as the limit of $\frac{\Delta Y}{\Delta x}$ as Δx approaches zero. This is as close to a particular x as we can achieve.

Rules for Differentiation:

Constants:

If : $Y = a$, where a is a constant.

then: $\frac{\partial Y}{\partial x} = 0$

Linear Functions:

If : $Y = a + bx$

then: $\frac{\partial Y}{\partial x} = b$ (note how this corresponds to the slope of a linear function)

Power Functions:

If: $Y = ax^b$

then: $\frac{\partial Y}{\partial x} = bax^{b-1}$

Sums and Differences:

If: $Y = u + w$

then: $\frac{\partial Y}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}$

If: $Y = u - w$

then: $\frac{\partial Y}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}$

Products:

If: $Y=u*w$

then: $\frac{\partial Y}{\partial x} = \frac{\partial u}{\partial x} * w + \frac{\partial w}{\partial x} * u$

Quotients:

If: $Y = \frac{u}{w}$

then: $\frac{\partial Y}{\partial x} = \frac{\frac{\partial u}{\partial x} * w - \frac{\partial w}{\partial x} * u}{w^2}$

Exponential base *e*:

If: $Y = e^{t}$ then: $\dfrac{\partial Y}{\partial t} = e^{t}$

If: $Y = Ae^{rt}$ then: $\dfrac{\partial Y}{\partial t} = Are^{rt}$

Natural Logarithms:

If: $Y = \ln t$ then: $\dfrac{\partial Y}{\partial t} = \dfrac{1}{t}$

If: $Y = \ln at$ then: $\dfrac{\partial Y}{\partial t} = \dfrac{1}{t}$

Partial Derivatives

$Y = f(x,z) = \mathrm{a}x^2z$

$\dfrac{\partial Y}{\partial x} = f_x = 2\mathrm{a}xz$, which assumes that z is constant

$\dfrac{\partial Y}{\partial z} = f_z = \mathrm{a}x^2$, which assumes that x is constant

The Differential

$\dfrac{dY}{dx} = f'(x)$ (derivative)

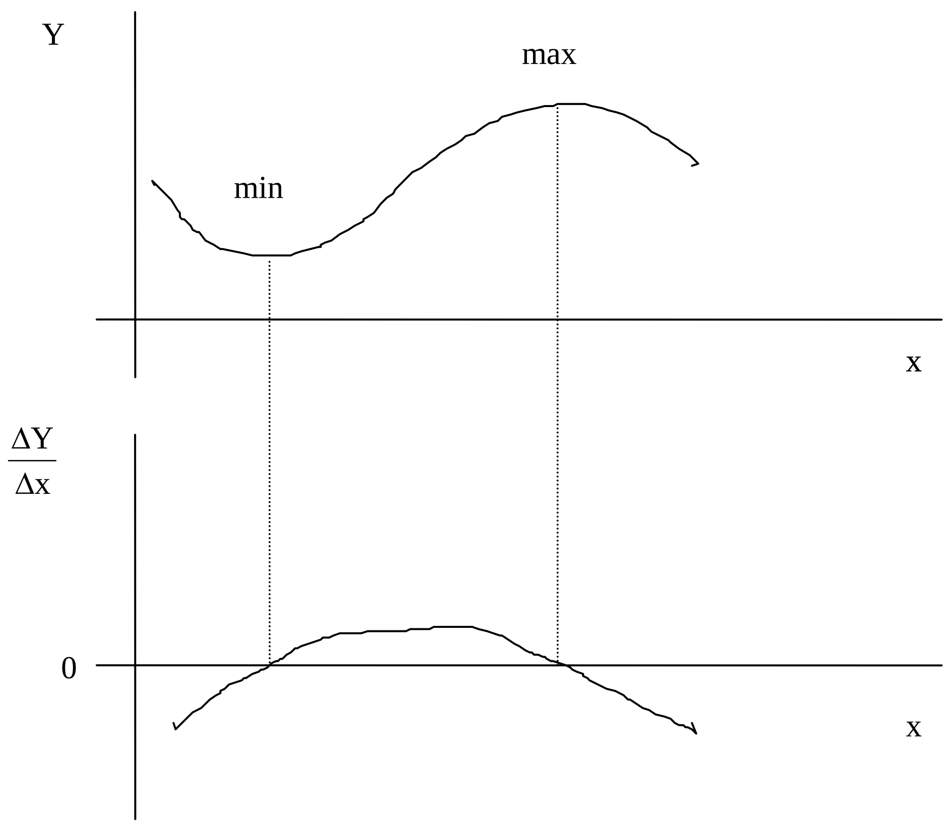
$dY = f'(x)dx$ (differential)

$dY = f_1(x,z)dx + f_2(x,z)dz$ (total differential of a multivariate function)

III. Optimization

A. Unconstrained Optimization

We are interested in finding the value of a control variable (i.e., the one that a decision maker can manipulate – *x* below) to maximize or minimize the dependent variable (i.e., the objective function). For example, a firm may want to know how many workers to hire to minimize the cost of producing a given output, or a firm may want to know the level of output that maximizes profits. When the dependent variable (objective function) is maximized or minimized, the slope of the function is equal to zero. Thus, taking the derivative of the function (i.e., the slope) and setting it equal to zero will allow us to find the optimal *x*. To know whether we have found a maximum or minimum requires us to look and the direction in which the slope is moving as x increases. This latter information is gleaned from the second derivative of the function.



<u>Example</u>:

$S = 10 + 5A - 1.5A^2$ (this function was taken from plotting past advertising vs. sales and fitting a function)

where:

S: total sales (millions of $)

A: advertising expenditures (millions of $)

Objective: Find the $ amount of advertising expenditures MAXIMIZES sales revenue?

Solution:

1) Take the first derivative (or slope of the objective function).

$$\frac{\partial S}{\partial A} = 5 - 3A$$

2) Set the slope equal to zero and solve for A (the control variable). We call this the first-order condition (FOC)

$$5 - 3A = 0$$
$$5 = 3A$$
$$A = \frac{5}{3}$$
$$A = \$1.67\text{million}$$

How do we know we are maximizing the objective function? We could plot the function or we could take the second derivative of the function. We call this the second-order condition (SOC).

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\partial(5 - 3A)}{\partial A} = -3$$

$$\frac{\partial^2 Y}{\partial x^2} < 0$$

This tells us that as A increases from its value determined in the FOC, the slope decreases. Therefore, we must have found a maximum when A = $1.67 million.

Note: The Second Derivative is the Slope of the Slope – direction the slope is moving as x increases.

We write this as $\frac{\partial^2 Y}{\partial x^2}$ or f_2 or f ”.

The second derivative is defined as:

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\partial\left(\frac{\partial Y}{\partial x}\right)}{\partial x}$$

If: $\frac{\partial^2 Y}{\partial x^2} < 0$ you have a maximum

If: $\frac{\partial^2 Y}{\partial x^2} > 0$ you have a minimum

<u>Optimizing Multivariate Functions:</u>

Often, there are more than one control variable that determines the objective function.

For example, assume

$$\Pi = f(Q_1, Q_2)$$

where:

Π is total profits
Q_1 is output of good 1
Q_2 is output of good 2

The objective function is given by,

$$\Pi = -20 + 100Q_1 + 80Q_2 - 10Q_1^2 - 10Q_2^2 - 5Q_1Q_2$$

<u>FOC:</u>

Q_1 $\quad 100 - 20Q_1 - 5Q_2 = 0 \qquad (1)$

Q_2 $\quad 80 - 20Q_2 - 5Q_1 = 0 \qquad (2)$

Solve them for Q_1 and Q_2 (2 equations with 2 unknowns):

from (1) $\quad \begin{aligned} 5Q_2 &= 100 - 20Q_1 \\ Q_2 &= 20 - 4Q_1 \end{aligned} \qquad (3)$

from (2) $\quad \begin{aligned} 5Q_1 &= 80 - 20Q_2 \\ Q_1 &= 16 - 4Q_2 \end{aligned} \qquad (4)$

Substituting (3) into (4) we obtain:

$$Q_1 = 16 - 4(20 - 4Q_1)$$
$$= 16 - 80 + 16Q_1$$
$$-15Q_1 = -64$$
$$Q_1 \cong 4.267$$

Substituting from Q_1 into (3) we obtain,

$$Q_2 = 20 - 4 * 4.267$$
$$Q_2 \cong 2.933$$

B. Constrained Optimization

In most applications, optimizing an objective function involves one or more constraints.

Example:

Maximize total sales across two regions subject to the constraint of an advertising budget. Your company hired some hotshot consultant that estimated the following regions specific relationships between sales revenue in regions and advertising expenditures:

$$S_1 = 10 + 5A_1 - 1.5A_1^2$$
$$S_2 = 12 + 4A_2 - 0.5A_2^2$$

where:

S_i refers to sales revenue in region "i" (millions of $)

A_i refers to advertising expenditures in region "i" (millions of $)

Additionally, you are provided with an advertising budget of $750,000 that cannot be exceeded. .

Objective: To allocate advertising expenditures so as to maximize sales.
Stated precisely,

$MAX(S_1 + S_2)$

Subject to (s.t.) $\quad A_1 + A_2 = 0.75$

i.e.,

MAX $\quad 22 + 5A_1 - 1.5A_1^2 + 4A_2 - 0.5A_2^2$

s.t. $\quad A_1 + A_2 = 0.75$

There are two general approaches:

i) Substitution approach
ii) Lagrangian approach

The substitution approach:

$$\text{MAX} \quad 22 + 5A_1 - 1.5A_1^2 + 4A_2 - 0.5A_2^2$$
$$\text{s.t.} \qquad A_1 + A_2 = 0.75$$

Note:

$$A_1 = 0.75 - A_2 \text{ (from the constraint)}$$

This allows the objective function to be rewritten as:

$$\text{MAX} \quad \begin{cases} 22 + 5*(0.75 - A_2) - 1.5*(0.75 - A_2)^2 + 4A_2 - 0.5A_2^2 \\ 22 + 3.75 - 5A_2 - 1.5*(0.5635 - 1.5A_2 + A_2^2) + 4A_2 - 0.5A_2^2 \\ 24.91 + 1.25A_2 - 2A_2^2 \end{cases}$$

$$\textit{wrt } A_2$$

<u>FOC</u>

$$1.25 - 4A_2 = 0$$
$$A_2 = 0.3125$$

From the constraint it follows that:

$$A_1 = 0.75 - A_2$$
$$A_1 = 0.75 - 0.3125$$
$$A_1 = 0.4375$$

Thus, you allocate advertising dollars over the two regions in the following manner:

$$A_1 = \$437{,}500$$
$$A_2 = \$312{,}500$$

NOTE:
A general rule is that the MARGINAL BENEFITS (MB refers to the sales associated with the last dollar spent) must be equal for optimization. i.e., $\left.\frac{\partial S_1}{\partial A_1}\right|_{A_1=0.4375} = \left.\frac{\partial S_2}{\partial A_2}\right|_{A_2=0.3125}$

We have:

$$\left.\frac{\partial S_1}{\partial A_1}\right|_{A_1=0.4375} = 5 - 3A_1 \qquad\qquad \left.\frac{\partial S_2}{\partial A_2}\right|_{A_2=0.3125} = 4 - A_2$$
$$= 5 - 3*(0.4375) \qquad\qquad = 4 - 0.3125$$
$$= 3.6875 \qquad\qquad = 3.6875$$

Lagrangian Approach:

Here we specify the constraint as part of the optimization function, called a Lagrangian function:

$$L = 22 + 5A_1 - 1.5A_1^2 + 4A_2 - 0.5A_2^2 + \lambda(0.75 - A_1 - A_2)$$

We then differentiate wrt each of the control variables and the constraint.

<u>FOC</u>

$$\begin{array}{lll} A_1 & 5 - 3A_1 - \lambda = 0 & (1) \\ A_2 & 4 - A_2 - \lambda = 0 & (2) \\ \lambda & 0.75 - A_1 - A_2 = 0 & (3) \end{array}$$

From (1) and (2) we solve for A_1 in terms of A_2,

$$5 - 3A_1 = 4 - A_2$$

$$3A_1 = 1 + A_2 \qquad (4)$$

$$A_1 = \frac{1}{3} + \frac{A_2}{3}$$

From (3) and (4) we solve for A_2,

$$0.75 - \left(\frac{1}{3} + \frac{A_2}{3}\right) - A_2 = 0$$

$$0.4167 - 1.333A_2 = 0$$

$$A_2 \cong 0.3125$$

And from (3) we solve for we solve for A_1,

$$A_1 = 0.333 + \frac{0.3125}{3}$$

$$A_1 = 0.4375$$

The value added of this approach is that we have identified something called the shadow price (λ) of the constraint.

From (2) we know that:

$$\lambda = 4 - A_2$$

$$= 4 - 0.3125$$

$$= 3.6875$$

This value is the marginal benefit (in terms of sales) of another dollar of advertising, in the case that constraint is loosened.